A LINEAR OPERATOR AND ITS APPLICATIONS TO CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

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I. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z:|z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}_p$ is said to be in the class $\mathcal{R}_p(\alpha)$ if it satisfies

$$\mathrm{Re}\{f^{(p)}(z)\} > \alpha \tag{1.2}$$

for some α $(\alpha < p!)$ and for all $z \in \mathcal{U}$. A function $f(z)$ belonging to the class $\mathcal{A}_p$ is said to be p-valently starlike of order α if and only if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.3}$$

for some α $(0 \leqq \alpha < p)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{S}_p^*(\alpha)$ the class of all functions in $\mathcal{A}_p$ which are p-valently starlike of order α in $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{A}_p$ is said to be p-valently convex of order α if and only if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

for some α $(0 \leqq \alpha < p)$ and for all $z \in \mathcal{U}$. Denoting by $\mathcal{C}_p(\alpha)$ the class of all functions in $\mathcal{A}_p$ which are p-valently convex of order α in $\mathcal{U}$, it is easily seen that

$$f(z) \in \mathcal{C}_p(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}_p^*(\alpha) \qquad (0 \leqq \alpha < p) \tag{1.5}$$

Further, a function $f(z)$ belonging to $\mathcal{A}_p$ is said to be p-valently close-to-convex of order α if and only if there exists a p-valently starlike function $g(z)$ such that

$$\mathrm{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \tag{1.6}$$

for some α ($0 \leqq \alpha < p$) and for all $z \in \mathcal{U}$.

For the functions $f_j(z)$ ($j = 1,2$) defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+p} z^{n+p} \qquad (p \in \mathbb{N}), \tag{1.7}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{1,n+p} a_{2,n+p} z^{n+p}. \tag{1.8}$$

For a function $f(z)$ belonging to the class $\mathcal{A}_p$, we define the generalized Libera integral operator $J_{c,p}$ by

$$J_{c,p}(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\,dt, \qquad c > -p. \tag{1.9}$$

For $p = 1$ and $c \in \mathbb{N}$, the operator $J_{c,1}$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some interesting results for the operator $J_{c,p}$ was proved by Saitoh [11] and Saitoh et al. [12].

Now, let the function $\phi_p(a,c)$ be defined by

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \qquad (z \in \mathcal{U}), \tag{1.10}$$

for $c \neq 0, -1, -2, \ldots$, where $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\cdots(a+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.11}$$

Also, we define a linear operator $L_p(a,c)$ on A_p by

$$L_p(a,c;z)f(z) = \left(\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \right) * f(z) \tag{1.12}$$

for $f(z) \in A_p$ and $c \neq 0, -1, -2, \ldots$.

The operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

$L_p(a,c)$ has the integral representation

$$L_p(a,c;z)f(z) = \int_0^1 u^{-p} f(uz)\, d\mu(u), \tag{1.13}$$

where μ satisfies

$$d\mu(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{B(a,c-a)}\, du ,$$

and

$$\int_0^1 d\mu(u) = 1.$$

REMARKS. (1) For $f(z) \in A_1 = A$,

$$L_1(n+1,1;z)f(z) = D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$$

is Ruscheweyh derivative of $f(z)$ ([8]).

(2) For $f(z) \in A_p$,

$$L_p(n+p,1;z)f(z) = D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$$

is Ruscheweyh derivative introduced by Goel and Sohi [3].

(3) For $f(z) \in \mathcal{A}_p$,

$$L_p(c+p, c+p+1; z)f(z) = J_{c,p}(f(z))$$

is the generalized Libera integral operator ([11], [12]).

(4) $\phi_1(a,c;z)$ is an incomplete beta function, related to the Gauss hypergeometric functions by

$$\phi_1(a,c;z) = z\,{}_2F_1(1,a;c;z).$$

2. Some results

We shall now prove the following results.

THEOREM 1. Let $f(z) \in \mathcal{A}_p$ and $c > a > 0$. If $f(z) \in \mathcal{R}_p(\alpha)$ $(z \in \mathcal{U}\,,\ \alpha < p!)$, then we have

$$L_p(a,c;z)f(z) \in \mathcal{R}_p(\alpha).$$

PROOF. It is sufficient to show that

$$\mathrm{Re}\left\{ \frac{d^p}{dz^p} L_p(a,c;z)f(z) \right\} > \alpha \qquad (z \in \mathcal{U}).$$

Using the integral representation, we have

$$\begin{aligned} \mathrm{Re}\left\{ \frac{d^p}{dz^p} L_p(a,c;z)f(z) \right\} &= \frac{1}{B(a,c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}\mathrm{Re}\{f^{(p)}(uz)\}du \\ &> \frac{\alpha}{B(a,c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}du \\ &= \frac{\alpha}{B(a,c-a)} B(a,c-a) \\ &= \alpha, \end{aligned}$$

which evidently completes the proof of Theorem 1.

COROLLARY 1. Let $f(z) \in \mathcal{A}_p$ and $c > -p$. If $f(z) \in \mathcal{R}_p(\alpha)$ $(z \in \mathcal{U},\ \alpha < p!)$, then we have

$$J_{c,p}(f(z)) \in \mathcal{R}_p(\alpha).$$

In order to prove our main results depicting properties of the function $L_p(a,c;z)f(z)$, we shall need the following lemma.

LEMMA. Let $\psi(z)$ and $g(z)$ be analytic in $\mathbb{U}$ and satisfy

$$\psi(0) = \psi'(0) = \dots = \psi^{(p-1)}(0) = 0, \ \psi^{(p)}(0) \neq 0,$$
$$g(0) = g'(0) = \dots = g^{(p-1)}(0) = 0, \ g^{(p)}(0) \neq 0.$$

Suppose that for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$) we have

$$\psi(z) * \frac{1 + \rho\sigma z}{1 - \sigma z} g(z) \neq 0 \qquad (z \in \mathbb{U}-\{0\}). \tag{2.1}$$

Then for each function $F(z)$ analytic in $\mathbb{U}$ and satisfying

$$\mathrm{Re}\{F(z)\} > 0 \qquad (z \in \mathbb{U}), \tag{2.2}$$

we have

$$\mathrm{Re}\left\{ \frac{(\psi * Fg)(z)}{(\psi * g)(z)} \right\} > 0 \qquad (z \in \mathbb{U}). \tag{2.3}$$

REMARK. In the case $p = 1$, this lemma was given by Ruscheweyh and Small [9]. The proof of this lemma is similar to their proof.

Applying the above lemma, we now have

THEOREM 2. Let $f(z) \in S_p^*(\alpha)$ $(0 \leqq \alpha < p)$ and let

$$L_p(a,c;z)\left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \qquad (z \in \mathbb{U}-\{0\}) \tag{2.4}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \dots$.
Then we have

$$L_p(a,c;z)f(z) \in S_p^*(\alpha).$$

PROOF. It is sufficient to show that

$$\mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)f(z)} \right\} > \alpha \tag{2.5}$$

for $z \in \mathbb{U}$. Since

$$\mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)f(z)} \right\} = \mathrm{Re}\left\{ \frac{L_p(a,c;z)zf'(z)}{L_p(a,c;z)f(z)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{\phi_p(a,c;z)*(zf'(z))}{\phi_p(a,c;z)*f(z)} \right\}, \tag{2.6}$$

putting

$$\psi(z) = \phi_p(a,c;z), \ F(z) = \frac{zf'(z)}{f(z)} - \alpha, \text{ and } g(z) = f(z)$$

in Lemma, we can see that

$$\mathrm{Re}\left\{ \frac{(\psi*Fg)(z)}{(\psi*g)(z)} \right\} = \mathrm{Re}\left\{ \frac{\phi_p(a,c;z)*\{zf'(z) - \alpha f(z)\}}{\phi_p(a,c;z)*f(z)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)f(z)} \right\} - \alpha > 0, \tag{2.7}$$

which completes the proof of Theorem 2.

COROLLARY 2. Let $f(z) \in S_p^*(\alpha)$ $(0 \leqq \alpha < p)$ and let

$$L_p(c+p, c+p+1; z)\left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in \mathbb{U} - \{0\}) \tag{2.8}$$

for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$, and for $c \neq -p-1, -p-2, -p-3, \ldots$. Then we have

$$J_{c,p}(f(z)) \in S_p^*(\alpha).$$

Setting $p = 1$ in Theorem 2, we have

COROLLARY 3 (Owa et al. [7]). Let $f(z) \in S_1^*(\alpha)$ $(0 \leqq \alpha < 1)$ and let

$$L_1(a,c;z)\left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \qquad (z \in \mathbb{U} - \{0\})$$

for each σ $(|\sigma| = 1)$ and ρ $(|\rho| = 1)$, and for $c \neq 0, -1, -2, \ldots$. Then we have

$$L_1(a,c;z)f(z) \in S_1^*(\alpha).$$

Next, we prove

THEOREM 3. Let $f(z) \in C_p(\alpha)$ $(0 \leqq \alpha < p)$ and let

$$L_p(a,c;z)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)zf'(z) \neq 0 \qquad (z \in \mathcal{U}-\{0\}) \tag{2.9}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \ldots$. Then we have

$$L_p(a,c;z)f(z) \in \mathcal{C}_p(\alpha).$$

PROOF. Note that $f(z) \in \mathcal{C}_p(\alpha)$ if and only if $zf'(z) \in \mathcal{S}_p^*(\alpha)$. By using Theorem 2, we know that

$$\begin{aligned} f(z) \in \mathcal{C}_p(\alpha) &\Longleftrightarrow zf'(z) \in \mathcal{S}_p^*(\alpha) \\ &\Longrightarrow L_p(a,c;z)zf'(z) \in \mathcal{S}_p^*(\alpha) \\ &\Longrightarrow z[L_p(a,c;z)f(z)]' \in \mathcal{S}_p^*(\alpha) \\ &\Longleftrightarrow L_p(a,c;z)f(z) \in \mathcal{C}_p(\alpha), \end{aligned} \tag{2.10}$$

which completes the proof of Theorem 3.

COROLLARY 4. Let $f(z) \in \mathcal{C}_p(\alpha)$ $(0 \leqq \alpha < p)$ and let

$$L_p(c+p,c+p+1;z)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)zf'(z) \neq 0 \qquad (z \in \mathcal{U}-\{0\}) \tag{2.11}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq -p-1, -p-2, -p-3, \ldots$. Then we have

$$J_{c,p}(f(z)) \in \mathcal{C}_p(\alpha).$$

Finally, we prove

THEOREM 4. Let $f(z) \in \mathcal{K}_p(\alpha)$, i.e., there exists $g(z) \in \mathcal{S}_p^*(0)$ such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathcal{U}).$$

Further, let

$$L_p(a,c;z)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)g(z) \neq 0 \qquad (z \in \mathcal{U}-\{0\}) \tag{2.12}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq 0, -1, -2, \ldots$.

Then we have

$$L_p(a,c;z)f(z) \in \mathcal{K}_p(\alpha). \tag{2.13}$$

PROOF. By Theorem 2, if $g(z) \in S_p^*(0)$, then $L_p(a,c;z)g(z) \in S_p^*(0)$. It is sufficient to show that

$$\mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)g(z)} \right\} > \alpha \tag{2.14}$$

for $z \in \mathbb{U}$. Since

$$\mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)g(z)} \right\} = \mathrm{Re}\left\{ \frac{L_p(a,c;z)zf'(z)}{L_p(a,c;z)g(z)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{\phi_p(a,c;z)*(zf'(z))}{\phi_p(a,c;z)g(z)} \right\}, \tag{2.15}$$

setting

$$\psi(z) = \phi_p(a,c;z),\ F(z) = \frac{zf'(z)}{g(z)} - \alpha, \text{ and } g(z) = g(z)$$

in Lemma, we observe that

$$\mathrm{Re}\left\{ \frac{(\psi*Fg)(z)}{(\psi*g)(z)} \right\} = \mathrm{Re}\left\{ \frac{\phi_p(a,c;z)*\{zf'(z) - \alpha g(z)\}}{\phi_p(a,c;z)*g(z)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{z[L_p(a,c;z)f(z)]'}{L_p(a,c;z)g(z)} \right\} - \alpha > 0, \tag{2.16}$$

which completes the proof of Theorem 4.

COROLLARY 5. Let $f(z) \in \mathcal{K}_p(\alpha)$, i.e., there exists $g(z) \in S_p^*(0)$ such that

$$\mathrm{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \qquad (0 \leqq \alpha < p;\ z \in \mathbb{U}).$$

Further, let

$$L_p(c+p,c+p+1;z)\left(\frac{1+\rho\sigma z}{1-\sigma z} \right)g(z) \neq 0 \qquad (z \in \mathbb{U}-\{0\}) \tag{2.17}$$

for each σ ($|\sigma| = 1$) and ρ ($|\rho| = 1$), and for $c \neq -p-1, -p-2, -p-3, \ldots$. Then we have

$$J_{c,p}(f(z)) \in \mathcal{K}_p(\alpha).$$

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